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Hermite–Pade Approximation to a Nikishin Type System of Meromorphic Functions

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Nikishin type systems of meromorphic functions whose poles lie symmetrically with respect to the real axis are considered. For such systems, it is shown that the main diagonal of the associated Hermite–Pade approximants converges and the poles are located by the zeros of the corresponding denominators. An interesting feature is that multipoint Pade approximation plays a key role in the proof. © 1995 Academic Press, Inc.

1. INTRODUCTION

1. Simultaneous rational approximation of finite systems of analytic functions has its origin in Hermite's investigations on the trascendence of e. Since then, other applications in number theory were obtained. After Hermite's works, mostly the formal aspect of the theory, which deals with algebraic relations and normality of such systems, was developed. Only recently, relatively large classes of systems of analytic functions have been found for which the convergence theory offers reasonable results. For references and more information concerning the formal and analytic theory see [1, 2] and the review paper [3].

In [1], E. M. Nikishin introduced an important class of such systems, which has been named after him. They are formed by a finite number of Cauchy transforms of finite positive Borel measures supported on the same interval which are internally linked.

Here, we deal with Nikishin systems of two functions perturbed with rational aggregates. The reason for considering this generalization is to study the effect that the poles of the approximated functions have on those of the rational simultaneous approximants and check certain phenomena which occur in Pade approximation (as, for example, attraction of poles). Although, for simplicity in the exposition and reading, we restrict our attention to the case of two functions, many of the results may be extended to the general situation of m functions. The main ideas in the

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proofs become transparent from the simple case, the rest of the ingredients may be found in [2].

2. Let Δ_1, Δ_2 be non-intersecting bounded intervals of the real line $\mathbb{R}$ (the intervals may be taken unbounded, but that restriction simplifies some arguments in the proof of Lemma 1 below, see [2, Lemma 3]); μ and σ denote finite positive Borel measures on Δ_1 and Δ_2, respectively, whose support (supp) contain an infinite set of points. For an arbitrary positive measure λ on a segment Δ, we write

$$\hat{\lambda}(z) = \int_{\Delta} \frac{d\lambda(x)}{z - x}.$$

Note that $\hat{\lambda}$ is holomorphic in $\mathbb{C} \setminus \Delta$, and takes real values on $\mathbb{R} \setminus \Delta$.

Consider fixed rational functions $r_i = s_i/t_i$, $i = 1, 2, 3$, where s_i and t_i are polynomials with real coefficients, $\deg t_i = d_i$, $\deg s_3 = d_4$, $r_3 \in L_1(\mu)$. Set

$$g_1(x) \equiv 1, \qquad g_2(x) = r_3(x)\hat{\sigma}(x), \qquad x \in \Delta_1.$$

In this paper, we consider the simultaneous approximation of the functions

$$f_i(z) = \int_{\Delta_1} \frac{g_i(x)}{z - x}\, d\mu(x) + r_i(z), \qquad i = 1, 2, \tag{1}$$

by means of interpolating rational functions (Hermite–Pade approximants). To be more specific, we study the convergence of the sequences $\pi_{n,i} = P_{n,i}/Q_n$, $n \in \mathbb{N}$, $i = 1, 2$, where $P_{n,i}$ and Q_n are polynomials chosen to satisfy the conditions:

(i) $\deg Q_n \le n$, $Q_n \neq 0$,

(ii) $(Q_n f_i - P_{n,i})(z) = O(z^{-n/2-1})$, $i = 1, 2$, for even n or $(Q_n f_i - P_{n,i})(z) = O(z^{-(n-1)/2-i})$, $i = 1, 2$, for odd n.

Note that for even n the interpolation conditions are equally distributed between the two functions, while for odd n the second function receives one more.

The existence of the indicated polynomials reduces to solving a system of n linear homogeneous equations on the $n + 1$ coefficients of Q_n. Therefore a non-trivial solution always exists. $P_{n,i}$ is taken as the polynomial part of $Q_n f_i$. In general, for fixed n (unlike the case of classical Pade approximants) the rational functions are not uniquely determined. Therefore, given n, we consider any fixed possible solution to (i)–(ii). In the sequel, we normalize Q_n to have leading coefficient equal to one.

In [1], E. M. Nikishin proved (when $r_i \equiv 0$, $i = 1, 2$ and $r_3 \equiv 1$) that

$$\pi_{n,i} \underset{n}{\rightrightarrows} f_i, \qquad K \subset D = \overline{\mathbb{C}} \setminus \Delta_i, \tag{2}$$

uniformly on each compact subset of D. In the same paper, the general concept of Nikishin system of m functions was introduced. For such systems, in [2] we proved that (2) takes place for all the m functions. In the meromorphic case, we obtain some convergence properties of type (2); but, in general, with a weaker topology (see Theorem 1 below).

3. Before stating our main result, we must introduce some notations. Given a compact set $K \subset \mathbb{C}$, by cap(K) we denote its (outer) logarithmic capacity. Let $\{\varphi_n\}$, $n \in \mathbb{N}$, and φ be continuous complex-valued functions defined on a region $G \subset \mathbb{C}$. We say that φ_n converges in capacity to φ on each compact subset K of G ($\varphi_n \xrightarrow[n]{\text{cap}} \varphi$, $K \subset G$), if for every $\varepsilon > 0$

$$\text{cap}\{z \in K : |\varphi_n - \varphi| \geq \varepsilon\} \underset{n}{\to} 0.$$

In the sequel, this is denoted by

$$\varphi_n \xrightarrow[n]{\text{cap}} \varphi, \qquad K \subset G.$$

We have:

Theorem 1. *For the functions defined above and* $i = 1, 2$

$$\pi_{n,i} \xrightarrow[n]{\text{cap}} f_i, \qquad K \subset D = \mathbb{C} \setminus \Delta_1. \tag{3}$$

Convergence in capacity is due to the fact that some of the poles of π_n may fall in D altering uniform convergence. Nevertheless, under additional assumptions, the number of these zeros coincides, for sufficiently large n, with the total amount of poles (counting their multiplicities) of f_1 and f_2 in that region. Then, convergence in capacity yields uniform convergence. Two such cases are considered in the following corollaries.

Corollary 1. *Assume that*:

(a) r_1 *and* r_2 *have no common finite poles, and all of them lie in* $\mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$;

(b) r_3 *has no zero or pole on* $\Delta_1 \cup \Delta_2$.

Then,

$$\pi_{n,i} \xrightarrow[n]{s} f_i, \qquad K \subset D, \qquad i = 1, 2.$$

This notation stands for uniform convergence on compact subsets of D in the spherical metrics. More precisely, we prove that, for all sufficiently large n, Q_n has exactly $n - (d_1 + d_2)$ simple zeros on Δ_1; the rest of the zeros of Q_n are "attracted" by the poles of f_1 and f_2, according to their multiplicities and in $D' = D \setminus \{z\colon (t_1 t_2)(z) = 0\}$,

$$\pi_{n,i} \underset{n}{\rightrightarrows} f_i, \qquad K \subset D', \qquad i = 1,2.$$

In particular, this implies that for large n, $\deg Q_n = n$; thus, for such indexes the rational functions $\pi_{n,i}$ are uniquely determined.

COROLLARY 2. *Assume that*:

(a′) $r_1 = r_2 + r_2'$, *where* r_2 *and* r_2' *have no common finite poles, and all of them lie in* $\mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$;

(b′) $r_3 \equiv 1$ *and* $\hat{\sigma} - 1$ *is either strictly positive or strictly negative on* $\mathbb{C} \setminus \Delta_2$.

Then, for even n's,

$$\pi_{n,i} \xrightarrow[n]{s} f_i, \qquad K \subset D.$$

Remark 1. Multipoint Pade approximants may be defined as usual (a.e. see [4]). Extensions of Theorem 1 and the Corollaries may be proved for such approximants and also for rational perturbations of Nikishin systems of m functions even when the measures are supported on unbounded sets (see [2]).

2. PROOF OF THEOREM 1

1. Let us first obtain some auxiliary relations. In the sequel $n' = n/2$ if n is even and $n' = (n-1)/2$ if n is odd. Also, we assume that $n > 2d$, $d = d_1 + d_2 + d_3 + d_4 + 2$. This condition on n is to guarantee that all forthcoming statements make sense; in fact, for each particular formula better lower bounds for n may be given but we only need these relations for large positive integers.

LEMMA 1. *Let* $h_{n,i}$, $i = 1,2$, *be arbitrary polynomials such that*

$$\begin{aligned} &\deg h_{n,i} \le n' - d_i - 1, \qquad i = 1,2, \text{ if } n \text{ is even} \\ &\deg h_{n,1} \le n' - d_1 - 1, \qquad \deg h_{n,2} \le n' - d_2, \qquad \text{for odd } n. \end{aligned} \tag{4}$$

Then

$$0 = \int_{\Delta_1} Q_n(x)\,[h_{n,2} t_2 r_3 \hat{\sigma} - h_{n,1} t_1](x)\, d\mu(x). \tag{5}$$

On the other hand, let $h_{n,i}$, $i = 1,2$, be arbitrary polynomials such that

$$\begin{array}{lll} \deg h_{n,i} \le n' - d_i, & i = 1,2, \text{ if } n \text{ is even} & \\ \deg h_{n,1} \le n' - d_1, & \deg h_{n,2} \le n' - d_2 + 1, & \text{for odd } n. \end{array} \tag{6}$$

Then

$$[h_{n,2}t_2F_{n,2} - h_{n,1}t_1F_{n,1}](z) = \int_{\Delta_1} Q_n(x)[h_{n,2}t_2r_3\hat{\sigma} - h_{n,1}t_1](x)\frac{d\mu(x)}{z-x}. \tag{7}$$

where $F_{n,i} = Q_nf_i - P_{n,i}$, $i = 1,2$.

Proof. Let $h_{n,i}$ be as in (4). From (ii) and Cauchy's Theorem

$$0 = \int_\Gamma (h_{n,i}t_i)(z)[Q_nf_i - P_{n,i}](z)\,dz = \int_\Gamma (Q_nh_{n,i}t_i)(z)\int_{\Delta_1}\frac{g_i(x)\,d\mu(x)}{z-x}\,dz,$$

where Γ is an arbitrary contour surrounding the segment Δ_1. Using Fubini's Theorem, Cauchy's integral formula and deleting the expression thus obtained for $i = 1$ from the one for $i = 2$ we arrive to (5).

Now, let $h_{n,i}$ be as in (6). From (ii), Cauchy's integral formula and Fubini's Theorem, we have

$$(h_{n,i}t_i)(z)[Q_nf_i - P_{n,i}](z) = \frac{1}{2\pi i}\int_\Gamma (Q_nh_{n,i}t_i)(\zeta)\int_{\Delta_1}\frac{g_i(x)\,d\mu(x)}{\zeta - x}\frac{d\zeta}{z-\zeta} = \int_{\Delta_1}[Q_nh_{n,i}t_ig_i](x)\frac{d\mu(x)}{z-x},$$

where Γ is an arbitrary contour surrounding the segment Δ_1 which leaves out the point z. Deleting the expression one obtains for $i = 1$ from the one for $i = 2$ you have (7). ∎

2. By construction, we know that $F_{n,i}$ has a certain amount of zeros at infinity. We will show that $F_{n,i}$ has an extra amount of zeros on Δ_2.

LEMMA 2. *For* $n > 2d$,

$$0 = \int_{\Delta_2} [PF_{n,1}t_1t_2s_3](x)\, d\sigma(x), \tag{8}$$

where P is an arbitrary polynomial, $\deg P \leq n' - d$. *In particular*, $F_{n,1}$ *has at least* $n' - 2d$ *zeros of odd multiplicity on* Δ_2.

Proof. Recall that $d = d_1 + d_2 + d_3 + d_4 + 2$. With P as above, taking in (5): $h_{n,2} = Pt_1t_3$, $h_{n,1} \equiv 0$; using Fubini's Theorem and (7), we find that

$$\begin{aligned} 0 &= \int_{\Delta_1} [PQ_nt_1t_2t_3r_3\hat{\sigma}](x)\, d\mu(x) \\ &= \int_{\Delta_2}\int_{\Delta_1} \frac{[PQ_nt_1t_2s_3](x)}{x - s}\, d\mu(x)\, d\sigma(x) \\ &= -\int_{\Delta_2} [PF_{n,1}t_1t_2s_3](s)\, d\sigma(s). \end{aligned}$$

Taking $P = P_1t_1t_2s_3$ in (8), with $\deg P_1 \leq n' - d - d_1 - d_2 - d_4$, we have

$$0 = \int_{\Delta_2} \left[P_1F_{n,1}t_1^2t_2^2s_3^2\right](x)\, d\sigma(x). \tag{9}$$

Assume that $F_{n,1}$ changes sign on Δ_2 at most $n' - 2d - 1$ times. Since $n' - 2d - 1 \leq n' - d - d_1 - d_2 - d_4$, we can construct a convenient polynomial P_1 such that

$$\left[P_1F_{n,1}t_1^2t_2^2s_3^2\right](x) \geq 0, \qquad x \in \Delta_2.$$

This contradicts (9). Therefore, $F_{n,1}$ has at least $n' - 2d$ zeros of odd multiplicity on Δ_2. ∎

In order to obtain a similar relation for the second function we need to introduce some more notation.

It is well known (see Appendix in [5]), that there exists a positive Borel measure σ_* on Δ_2 and a first degree polynomial $\mathscr{L}$ such that

$$\left(\int (z - x)^{-1}\, d\sigma(x)\right)^{-1} = \int (z - x)^{-1}\, d\sigma_*(x) + \mathscr{L}(z). \tag{10}$$

LEMMA 3. *For $n > 2d$,*

$$0 = \int_{\Delta_2} [PF_{n,2}t_1t_2s_3](x)\, d\sigma_*(x),$$

where P is an arbitrary polynomial, $\deg P \le n' - d$. In particular, $F_{n,2}$ has at least $n' - 2d$ zeros of odd multiplicity on Δ_2.

Proof. With P as above, taking in (5): $h_{n,2} = Pt_1t_3\mathcal{L}$, $h_{n,1} = Pt_2t_3r_3$; we find that

$$0 = \int_{\Delta_1} [PQ_nt_1t_2t_3r_3(\mathcal{L}\hat{\sigma} - 1)](x)\, d\mu(x).$$

From this, using (7), (10), and Fubini's Theorem, we obtain

$$\begin{aligned} 0 &= \int_{\Delta_1} [PQ_nt_1t_2t_3r_3\hat{\sigma}\hat{\sigma}_*](x)\, d\mu(x) \\ &= \int_{\Delta_2}\int_{\Delta_1} \frac{[PQ_nt_1t_2t_3g_2](x)}{x - s}\, d\mu(x)\, d\sigma_*(x) \\ &= -\int_{\Delta_2} [PF_{n,2}t_1t_2t_3](s)\, d\sigma_*(s). \end{aligned}$$

The statement concerning the zeros of $F_{n,2}$ on Δ_2 follows using the same arguments as in Lemma 2. ∎

3. *In order to complete the proof of Theorem 1, we need one more ingredient which is provided by [2, Lemma 2]. For the reader's convenience we include the corresponding statement, but first some notation.*

Let λ be a finite positive Borel measure on $\Delta \subset \mathbb{R}_+$, whose support contains infinitely many points,

$$\hat{\lambda}(z) = \int (z - x)^{-1}\, d\lambda(x), \qquad c_\nu = \int x^\nu\, d\lambda(x),$$

and $r = p/q$ ($\deg q = d$, $(p, q) \equiv 1$) be a rational function whose poles belong to $\overline{\mathbb{C}} \setminus \Delta$. Set $f = \hat{\lambda} + r$. Assume that $f = O(z^k)$ (as $z \to -\infty$, $z < 0$) $k \in \mathbb{Z}$. Fix an arbitrary integer $\kappa \ge k$. Consider a sequence of polynomials $\omega = \{\omega_m\}$, $m \in \Lambda \subset \mathbb{N}$, such that $\deg \omega_m = \kappa_m \le 2m + \kappa + 1$, whose zeros lie in $(-\infty, a]/[r = \infty]$, $a < 0$. Let $\{R_m\}$, $m \in \Lambda$, be any sequence of rational functions $R_m = p_m/q_m$ with real coefficients satisfying that for each m:

(i′) $\deg p_m \le m + \kappa$, $\deg q_m \le m$, $q_m \not\equiv 0$;
(ii″) $(q_mf - p_m)/\omega_m = O(z^{-m-1+\ell}) \in H(\mathbb{C} \setminus (\Delta \cup [r = \infty]))$,

where $\ell \in \mathbb{Z}_+$ is fixed.

We remark that for each m, there always exists R_m with real coefficients satisfying (i)–(ii) but in general it is not unique as in the case when $\ell = 0$.

LEMMA 4. *Assume that* $\{R_m\}$, $m \in \Lambda$, *with real coefficients satisfies* (i)–(ii) *and either the number of zeros of* ω_m *lying on a bounded segment of* $\mathbb{R}_-$ *tends to infinity as* $m \to \infty$ *or*

$$\sum_{\nu \geq 1} (c_\nu)^{-1/2\nu} = \infty.$$

Then

$$R_m \xrightarrow[m]{\text{cap}} f, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta, \qquad m \in \Lambda.$$

4. PROOF OF THEOREM 1. Let $\omega_{n,i}$, $i = 1, 2$, be a monic polynomial with $n' - 2d$ simple zeros at those points where $F_{n,i}$ changes sign on Δ_2, and $\deg \omega_{n,i} \geq n' - 2d$. This is possible according to Lemmas 1 and 2. Thus

$$(Q_n f_i - P_{n,i})/\omega_{n,i} = O(z^{-n-1+2d}) \in H(\mathbb{C} \setminus (\Delta_1 \cup [r_i = \infty])), \quad (11)$$

and condition (ii″) takes place (with $n = m$). Conditions (i′) are easy to verify with $\kappa = \max(0, \deg s_i - \deg t_i)$. Since Δ_2 is a bounded interval of the real line (the proof may be reduced without loss of generality to the case when $\Delta_2 \in \mathbb{R}_-$) Theorem 1 follows at once from Lemma 4. ▌

Remark 2. In this paper we have only considered the main diagonal of Hermite-Pade approximants. It is easy to verify that for sequences near the main diagonal (when n_i interpolation conditions are assigned to function f_i at infinity, with $n_1 + n_2 = n$, $|n_1 - n_2| \leq C$ independent of n) convergence in capacity also takes place. If Δ_1 is allowed to be unbounded then a Carleman type condition on the moments of μ must be required in order to use Lemma 4.

3. PROOF OF COROLLARIES

1. From (11) using, as in Lemma 1, Cauchy's Theorem we can obtain that Q_n is orthogonal with respect to $t_i g_i \, d\mu/\omega_{n,i}$, $i = 1, 2$, for all powers from 0 to $n - 3d - 1$. This implies that Q_n has at least $n - 3d$ zeros on Δ_1. But this is not enough to have uniform convergence. With additional restrictions, we can obtain a better estimate of the amount of zeros of Q_n lying on Δ_1.

LEMMA 5. *Under the assumptions of Corollary* 1, *there exists* $n_0 \in \mathbb{N}$, *such that for* $n > n_0$, Q_n *has at least* $n - d_1 - d_2$ *changes of sign on* Δ_1 (*zeros of odd multiplicity*).

Proof. Assume the contrary; that is, let Λ be an infinite set of indexes ($\Lambda \subset \mathbb{N}$) such that for $n \in \Lambda$, Q_n changes its sign on Δ_1 at most $n - d_1 - d_2 - 1$ times.

Let $\Lambda' \subset \Lambda$ be the set of even indexes in Λ and assume that Λ' has infinitely many points. We consider two cases; the first when $\deg s_3 \leq \deg t_3 + 1$ and the second when $\deg s_3 > \deg t_3 + 1$. In either cases we arrive to a contradiction.

For the first case, let us rewrite (5) in the following fashion:

$$0 = \int_{\Delta_1} Q_n(x)\left[h_{n,2} t_1^{-1} t_2 r_3 \hat{\sigma} - h_{n,1}\right](x) t_1(x)\, d\mu(x). \tag{12}$$

Note that $t_1^{-1} t_2 r_3 \hat{\sigma} = O(z^{d_2 - d_1})$ can be expressed as $\hat{\lambda} + r$, where $d\lambda(s) = (t_1^{-1} t_2 r_3)(s)\, d\sigma(s)$ is a constant signed measure on Δ_2, $r = p/q$ is the rational function (p and q are mutually prime polynomials)

$$r(x) = \int_{\Delta_2} \frac{(t_2 s_3)(x)(t_1 t_3)(s) - (t_2 s_3)(s)(t_1 t_3)(x)}{(x - s)(t_1 t_3)(s)(t_1 t_3)(x)}\, d\sigma(s),$$

and $\hat{\lambda} + r$ has at infinity either a pole of order $d_2 - d_1$ if $d_2 > d_1$ or a zero of degree $d_1 - d_2$ if $d_1 \geq d_2$ (recall that $\deg s_3 \leq \deg t_3 + 1$).

Set $m = n' - d_2 - 1$, $n \in \Lambda'$, and ω_m equal to the monic polynomial whose simple zeros are those points on Δ_1, where Q_n changes sign. Take q_m and p_m as the polynomials defined by the conditions (i')–(ii''), with $f = \hat{\lambda} + r$, $\kappa = d_2 - d_1$, and $\ell = 0$.

According to Lemma 4

$$R_m \xrightarrow[m]{\text{cap}} f, \qquad K \subset \mathbb{C} \setminus \Delta, \qquad m \in \Lambda'. \tag{13}$$

On the other hand, (see [2, Sect. 3, (1)]) it is easy to verify that

$$0 = \int_{\Delta_2} s^\nu q_m(s) q(s) \frac{d\lambda(s)}{\omega_m(s)}, \qquad \nu = 0, 1, \ldots, m - c - 1, \tag{14}$$

where c is the number of poles of r in $\mathbb{C} \setminus \Delta_2$. Relation (14) yields that, q_m has at least $m - c$ zeros on Δ_2. Therefore, the number of poles of R_m in $\overline{\mathbb{C}} \setminus \Delta_2$ is not greater than the number of poles of f in that region, and using [6, Lemma 1] we obtain

$$R_m(z) \xrightarrow[m]{s} f, \qquad K \subset \overline{\mathbb{C}} \setminus \Delta, \qquad m \in \Lambda'.$$

In particular, we have that for all sufficiently large $m \in \Lambda'$, $\deg q_m = m$, q_m has exactly $m - c$ zeros on Δ_2, and each pole of r attracts as many zeros of q_m as its order.

An integral expression for $[q_m t_1^{-1} t_2 r_3 \hat{\sigma} - p_m](x)$ is easy to obtain (see [2, Sect. 3, (2)]),

$$\left[q_m t_1^{-1} t_2 r_3 \hat{\sigma} - p_m\right](x) = \frac{\omega_m(x)}{(l_m q)(x)} \int_{\Delta_2} \frac{(q_m l_m q)(s)}{\omega_m(s)} \frac{d\lambda(s)}{x - s}, \quad (15)$$

where l_m denotes an arbitrary polynomial of degree $\leq m - c$. Note that from the restrictions on the degrees of q_m and p_m (conditions (i') applied to this case), we can take in (12): $h_{n,2} = q_m$, $P_m = 2$. Substituting (15) in (12) we find

$$0 = \int_{\Delta_1} \frac{(\omega_m Q_n)(x)}{(l_m q)(x)} \int_{\Delta_2} \frac{(q_m l_m q)(s)}{\omega_m(s)} \frac{d\lambda(s)}{x - s} t_1(x)\, d\mu(x). \quad (16)$$

Take in (16), l_m equals to the monic polynomial whose zeros are the zeros of q_m on Δ_2 $(n \geq n_0)$. The measure $d\lambda(s)$ has constant sign; thus considering the choice of l_m and ω_m, we have that the function standing under the outer integral has constant sign. Therefore, that integral cannot be equal to zero. Hence, Λ does not contain an infinite set of even indexes when $\deg s_3 \leq \deg t_3 + 1$.

If $\deg s_3 > \deg t_3 + 1$, then we rewrite (5) as

$$0 = \int_{\Delta_1} Q_n(x)\left[h_{n,2} - h_{n,1}(t_2 r_3 \hat{\sigma})^{-1} t_1\right](x)(t_2 r_3 \hat{\sigma})(x)\, d\mu(x).$$

From (10) we know that $\hat{\sigma}^{-1}$ is a Markov type function plus a first degree polynomial. Therefore, proceeding as above, there exist a positive Borel measure λ' on Δ_2 and a rational function r', such that

$$\left[(t_2 r_3 \hat{\sigma})^{-1} t_1\right](z) = \hat{\lambda}'(z) + r'(z),$$

where $\hat{\lambda}'(z) + r'(z)$ has at infinity either a pole of order $d_1 - d_2$ if $d_1 > d_2$ or a zero of degree $d_2 - d_1$ if $d_2 \geq d_1$. Reasoning as above, we conclude that when $\deg s_3 > \deg t_3 + 1$, Λ cannot have an infinite set of even indexes either.

For odd indexes, the proof follows the same guidelines. The main trick consists in taking each time q_m and p_m conveniently. Thus we conclude with Lemma 5. ∎

2. Proof of Corollary 1. From Theorem 1 (see (3))

$$\pi_{n,1} + \pi_{n,2} \xrightarrow[n]{\text{cap}} f_1 + f_2, \qquad K \subset D = \mathbb{C} \setminus \Delta_1. \tag{17}$$

On the other hand, from Lemma 5 we know that $\pi_{n,1} + \pi_{n,2}$ has for $n \geq n_0$ at most $d_1 + d_2$ poles in D, while $f_1 + f_2$ has in that region exactly $d_1 + d_2$ poles. Therefore, from (17) and [6, Lemma 1]

$$\pi_{n,1} + \pi_{n,2} \xrightarrow[n]{s} f_1 + f_2, \qquad K \subset D.$$

This and (3) immediately render

$$\pi_{n,i} \xrightarrow[n]{s} f_i, \qquad K \subset D, \qquad i = 1,2.$$

The proof of Corollary 1 is complete. ∎

3. *Proof of Corollary 2. This case may be reduced to the previous one. If* $\hat{\sigma} < 1$ *on* Δ_1, *we can take* $F_1 = f_1 - f_2$, $F_2 = f_2$. *Note that*

$$F_1(z) = \int_{\Delta_1} \frac{1 - \hat{\sigma}(x)}{z - x}\, d\mu(x) + r_2'(z),$$

and

$$F_2(z) = \int_{\Delta_1} \frac{1 - \hat{\sigma}(x)}{z - x}\, [\hat{\sigma}(x)/(1 - \hat{\sigma}(x))]\, d\mu(x) + r_2(z).$$

On the other hand, since $\hat{\sigma} < 1$, *it is easy to prove (see* [5, *Appendix*]) *that there exists a positive measure* σ_* *on* Δ_2 *such that*

$$\hat{\sigma}(z)/(1 - \hat{\sigma}(z)) = \hat{\sigma}_*(x).$$

Therefore, you can use Corollary 1 *for the functions* F_1, F_2. *After this you return to the initial functions obtaining the statement of Corollary* 1. *If* $\hat{\sigma} > 1$ *on* Δ_2, *the proof is analogous concluding with Corollary* 2. ∎

The reason for restricting our attention in Corollary 2 to even indexes n is to ensure that the denominators of the simultaneous Pade approximants corresponding to the systems of functions (f_1, f_2) and (F_1, F_2) coincide; otherwise (for odd n) in taking the difference, we lose one interpolation condition at infinity for the second function.

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